13.2

Limits of Quotients

Find the limits in Exercises 13–24 by rewriting the fractions first.

20. $$\lim_{\substack{(x,y)\to(4,3) \\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$$

Sol'n:

$$\frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}\cdot\frac{\sqrt{x}+\sqrt{y+1}}{\sqrt{x}+\sqrt{y+1}} = \frac{\cancel{x-y-1}}{\cancel{(x-y-1)}(\sqrt{x}+\sqrt{y+1})} = \frac{1}{\sqrt{x}+\sqrt{y+1}}$$

So

$$\lim_{\substack{(x,y)\to(4,3) \\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim_{\substack{(x,y)\to(4,3) \\ x\neq y+1}} \frac{1}{\sqrt{x}+\sqrt{y+1}} = \frac{1}{\sqrt{4}+\sqrt{3+1}} = \boxed{\frac{1}{4}}$$

13.2

Limits with Three Variables

Find the limits in Exercises 25–30.

25. $\lim_{P \to (1,3,4)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$

26. $\lim_{P \to (1,-1,-1)} \frac{2xy + yz}{x^2 + z^2}$

27. $\lim_{P \to (\pi,\pi,0)} (\sin^2 x + \cos^2 y + \sec^2 z)$

28. $\lim_{P \to (-1/4,\pi/2,2)} \tan^{-1} xyz$

29. $\lim_{P \to (\pi,0,3)} ze^{-2y} \cos 2x$

Sol'n: $\lim_{P \to (-\frac{1}{4}, \frac{\pi}{2}, 2)} \tan^{-1}(xyz) = \tan^{-1}\left(-\frac{1}{4} \cdot \frac{\pi}{2} \cdot 2\right) = \boxed{\tan^{-1}\left(-\frac{\pi}{4}\right)}$

13.2

Continuity for Three Variables

At what points (x, y, z) in space are the functions in Exercises 35–40 continuous?

38. a. $h(x, y, z) = \dfrac{1}{|y| + |z|}$ **b.** $h(x, y, z) = \dfrac{1}{|xy| + |z|}$

Sol'n: a) Note that h is not well-defined when $|y| = |z| = 0 \Rightarrow y = z = 0$.

Show using ε-δ def'n of continuity that h is continuous on $\{(x,y,z) \in \mathbb{R}^3 : y, z \neq 0\}$.

Let $(x_0, y_0, z_0) \in \mathbb{R}^3$ s.t. $y_0 \neq 0$, $z_0 \neq 0$. Let $\varepsilon > 0$. Then

$$|h(x,y,z) - h(x_0,y_0,z_0)| = \left| \frac{1}{|y| + |z|} - \frac{1}{|y_0| + |z_0|} \right|$$

$$= \left| \frac{|y_0| + |z_0| - |y| - |z|}{(|y| + |z|)(|y_0| + |z_0|)} \right|$$

$$\leq \frac{\big||y|-|y_0|\big|}{(|y|+|z|)(|y_0|+|z_0|)} + \frac{\big||z|-|z_0|\big|}{(|y|+|z|)(|y_0|+|z_0|)}. \quad (*)$$

Since $y_0 \neq 0$, $z_0 \neq 0$, $m := \min\{|y_0|, |z_0|\} > 0$. Then for $|y|, |z| \leq m$,

$$0 < 4m^2 = (2m)(2m) \leq (|y|+|z|)(|y_0|+|z_0|).$$

So $(*)$ becomes

$$|h(x,y,z) - h(x_0,y_0,z_0)| \leq \frac{\big||y|-|y_0|\big|}{4m^2} + \frac{\big||z|-|z_0|\big|}{4m^2}.$$

Then for $\delta = 2m^2\varepsilon$ with $\sqrt{|x-x_0|^2 + |y-y_0|^2 + |z-z_0|^2} < \delta$, we have

$$|h(x,y,z) - h(x_0,y_0,z_0)| \leq \frac{\big||y|-|y_0|\big|}{4m^2} + \frac{\big||z|-|z_0|\big|}{4m^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

And so h is continuous on the set $\{(x,y,z) \in \mathbb{R}^3 : y, z \neq 0\}$, i.e. $\mathbb{R}^3 \setminus x\text{-axis}$.

b) Similarly, one can show that $\frac{1}{|xy|+|z|}$ is continuous on the set

$\{(x,y,z) \in \mathbb{R}^3 : xy \neq 0 \text{ and } z \neq 0\}$ i.e. on $\mathbb{R}^3 \setminus (x\text{-axis} \cup y\text{-axis})$.

B.2

60. Does knowing that

$$2|xy| - \frac{x^2y^2}{6} < 4 - 4\cos\sqrt{|xy|} < 2|xy|$$

tell you anything about

$$\lim_{(x,y)\to(0,0)} \frac{4 - 4\cos\sqrt{|xy|}}{|xy|}?$$

Give reasons for your answer.

Sol'n: Yes: We have

$$\lim_{(x,y)\to(0,0)} \frac{4-4\cos\sqrt{|xy|}}{|xy|} \le \lim_{(x,y)\to(0,0)} \frac{2\cancel{|xy|}}{\cancel{|xy|}} = 2, \text{ and}$$

$$\lim_{(x,y)\to(0,0)} \frac{4-4\cos\sqrt{|xy|}}{|xy|} \ge \lim_{(x,y)\to(0,0)} \frac{2|xy| - \frac{x^2y^2}{6}}{|xy|} = \lim_{(x,y)\to(0,0)} 2 - \frac{\sqrt{x^2y^2}}{6}$$

$$= 2.$$

So we can conclude by the Squeeze Theorem that

$$\lim_{(x,y)\to(0,0)} \frac{4-4\cos\sqrt{|xy|}}{|xy|} = 2.$$

13.2

In Exercises 65–70, find the limit of f as $(x, y) \to (0,0)$ or show that the limit does not exist.

65. $f(x, y) = \dfrac{x^3 - xy^2}{x^2 + y^2}$

66. $f(x, y) = \cos\left(\dfrac{x^3 - y^3}{x^2 + y^2}\right)$

Sol'n: Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$.

Then $f(x,y) = \cos\left(\dfrac{x^3-y^3}{x^2+y^2}\right) = \cos\left(\dfrac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2}\right)$

$= \cos\left(r(\cos^3\theta - \sin^3\theta)\right) = f(r,\theta)$

and $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{r\to 0} f(r,\theta) = \lim\limits_{r\to 0} \cos\left(r(\cos^3\theta - \sin^3\theta)\right)$

$= \cos(0) = \boxed{1}$

13.2

In Exercises 71 and 72, define $f(0,0)$ in a way that extends f to be continuous at the origin.

71. $f(x,y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$

72. $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$

Sol'n: Polar coordinates: $x^2+y^2=r^2$, $x=r\cos\theta$, $y=r\sin\theta$.

Then $f(x,y) = \dfrac{3x^2y}{x^2+y^2} = \dfrac{3r^2\cos^2\theta\, r\sin\theta}{r^2} = 3r\cos^2\theta\sin\theta = f(r,\theta)$,

and

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta) = \lim_{r\to 0} 3r\cos^2\theta\sin\theta = 0.$$

So defining $f(0,0)=0$ extends f so that it is continuous at the origin. //.

13.3

Calculating First-Order Partial Derivatives

In Exercises 1–22, find $\partial f/\partial x$ and $\partial f/\partial y$.

22. $f(x, y) = \sum_{n=0}^{\infty} (xy)^n \quad (|xy| < 1)$

Sol'n:

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}(xy)^n\right) \underset{\uparrow}{=} \sum_{n=0}^{\infty} \frac{\partial}{\partial x}(xy)^n = \sum_{n=1}^{\infty} n(xy)^{n-1}\frac{\partial}{\partial x}(xy)$$

since $|xy|<1$

$$= \sum_{n=1}^{\infty} n(xy)^{n-1} y = \sum_{n=1}^{\infty} n x^{n-1} y^n$$

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\left(\sum_{n=0}^{\infty}(xy)^n\right) \underset{\uparrow}{=} \sum_{n=0}^{\infty} \frac{\partial}{\partial y}(xy)^n = \sum_{n=1}^{\infty} n(xy)^{n-1}\frac{\partial}{\partial y}(xy)$$

$|xy|<1$

$$= \sum_{n=1}^{\infty} n(xy)^{n-1} x. = \sum_{n=1}^{\infty} n x^n y^{n-1}$$

Alternatively: since $|xy|<1$,

$$f(xy) = \sum_{n=0}^{\infty} (xy)^n = \frac{1}{1-xy}.$$

So
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{1-xy}\right) = \frac{(1-xy)(0) - 1(-y)}{(1-xy)^2} = \frac{y}{(1-xy)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{1-xy}\right) = \frac{(1-xy)(0) - 1(-x)}{(1-xy)^2} = \frac{x}{(1-xy)^2}$$

13.3

Calculating Second-Order Partial Derivatives

Find all the second-order partial derivatives of the functions in Exercises 41–54.

41. $f(x, y) = x + y + xy$ 42. $f(x, y) = \sin xy$

43. $g(x, y) = x^2y + \cos y + y \sin x$

44. $h(x, y) = xe^y + y + 1$ 45. $r(x, y) = \ln(x + y)$

46. $s(x, y) = \arctan(y/x)$ 47. $w = x^2 \tan(xy)$

Sol'n:

$$s_x = \frac{\partial}{\partial x} s(x,y) = \frac{\partial}{\partial x} \arctan\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{-yx^{-2}}{1+\left(\frac{y}{x}\right)^2} = \frac{-y}{x^2+y^2}$$

$$s_y = \frac{\partial}{\partial y} s(x,y) = \frac{\partial}{\partial y} \arctan\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{x^{-1}}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2}$$

$$s_{xx} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(0)-(-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$s_{yy} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)(0)-(x)(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$S_{xy} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$S_{yx} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

13.3

Mixed Partial Derivatives

In Exercises 55–60, verify that $w_{xy} = w_{yx}$.

55. $w = \ln(2x + 3y)$ **56.** $w = e^x + x\ln y + y\ln x$

Sol'n: $w_x = e^x + \ln y + \frac{y}{x}$

$w_y = \frac{x}{y} + \ln x$

$w_{xy} = \frac{\partial}{\partial y} w_x = \frac{\partial}{\partial y}\left(e^x + \ln y + \frac{y}{x}\right) = \frac{1}{y} + \frac{1}{x}.$

$w_{yx} = \frac{\partial}{\partial x} w_y = \frac{\partial}{\partial x}\left(\frac{x}{y} + \ln x\right) = \frac{1}{y} + \frac{1}{x}.$

So indeed $w_{xy} = w_{yx}$.

13.3 **Using the Partial Derivative Definition**

In Exercises 63–66, use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.

66. $f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0), \end{cases}$

$\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(0,0)$

Sol'n: $$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\sin(h^3+0^4)}{h^2+0^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin(h^3)}{h^2}}{h} = \lim_{h\to 0} \frac{\sin(h^3)}{h^3} \underset{\text{L'Hôpital's}}{=} \lim_{h\to 0} \frac{\cancel{3h^2}\cos(h^3)}{\cancel{3h^2}} = 1$$

So $$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 1$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{\;\frac{\sin(h^4)}{h^2}\;}{h}$$

$$= \lim_{h\to 0}\frac{\sin(h^4)}{h^3} \underset{\text{L'Hôpital's}}{=} \lim_{h\to 0}\frac{4h^3\cos(h^4)}{3h^2} = \lim_{h\to 0}\frac{4}{3}h\cos(h^4)$$

$$= 0.$$

So $\left.\dfrac{\partial f}{\partial y}\right|_{(0,0)} = 0.$